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Ω-Expansion of van Kampen through Functional Integrals

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We give a functional integral version of the van Kampen Ω-expansion method for the master equation. Explicit expressions are given for the generating functional of correlation functions.

KEY WORDS: Correlation functions; functional integration; Markov processes; master equation; Ω-expansion.

1. INTRODUCTION

We give here a version of van Kampen's Ω-expansion[(1)] for the master equation using the formalism of functional integration. The method of van Kampen gives expansions in $\eta = \Omega^{-1}$ for the correlation functions of the Markov process defined by a master equation in which the transition probabilities have a canonical form.[(2)] The parameter Ω is usually the volume of the system, and the physics of the expansion is discussed in refs. 1 and 2. Our aim here is to use functional integral techniques to give closed expressions for the whole formal series in powers of η of the generating functional of correlation functions. This is accomplished in Section 2, where the formalism is presented in the case of one variable using the methods developed in refs. 3 and 4. Closed formulas are presented for the time-dependent regime and for the stationary state. In Section 3 we use the previous results in an explicit example. Appendix A presents the calculation of the "free" generating functional and Appendix B generalizes the results of Section 2 to master equations with several variables. The expansion

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discussed here is the analog of the usual loop expansion in power of $\hbar$ in quantum theory and η plays the role of $\hbar$ (see, for example, ref. 5).

The present version of the Ω-expansion has the advantage that the formal structure is displayed explicitly and that the terms of any order can be obtained in a systematic way through algebraic manipulations only from the basic expansion of the generating functional.

2. FUNCTIONAL INTEGRAL FORMALISM AND THE Ω-EXPANSION

The master equation for a variable $X(t)$ taking discrete values is

$$\frac{\partial}{\partial t} P(X, t \mid X', t') = \sum_{X''} [W(X \mid X'') P(X'', t \mid X', t') - W(X'' \mid X) P(X, t \mid X', t')] \tag{1}$$

where P is the conditional probability and $W(X \mid X')$ is the transition probability per unit time. In many situations of physical interest it is possible to introduce a parameter Ω representing the size of the system (we shall refer to Ω as the volume), and the transition probability $W(X \mid X')$ of the extensive variable X has the canonical form[2] [$f(\Omega)$ is a function of Ω, $r = X - X'$]

$$W(X \mid X') = f(\Omega)\, \omega\left(\frac{X'}{\Omega}, X - X'\right) = f(\Omega)\, \omega(x', r), \quad x' = \frac{X'}{\Omega} \tag{2}$$

where x' is intensive. The parameter $\eta = 1/\Omega$ is small and will be used as an espansion parameter. We allow $\omega(x, r)$ to depend on η in the form

$$\omega(x, r) = \omega_0(x, r) + \eta\omega_1(x, r) + \eta^2\omega_2(x, r) + \cdots$$

An expansion in powers of η for correlation functions (the Ω-expansion of van Kampen) is possible when W has the canonical form (2). Using (2) and putting $p(x, t \mid \cdots) = P(X, t \mid \cdots)$, we can write Eq. (1) as[6] ($\partial = \partial/\partial x$)

$$\eta \frac{\partial}{\partial t} p(x, t \mid \cdots) = \sum_{s \geqslant 0} \eta^s \sum_r (e^{-\eta r \partial} - 1)\, \omega_s(x, r)\, p(x, t \mid \cdots) \tag{3}$$

where we consider the intensive variable $x = X/\Omega$ as continuous, since Ω is very big. We assume now that x takes values in the whole real line. If X is initially restricted to be positive (for example, it it is a concentration of some chemical species), we can still consider that (3) is valid in the whole real axis if we have natural boundary conditions for (1) (see ref. 2 for a discussion). We can write (3) as

$$\eta \frac{\partial}{\partial t} p(x, t \mid \cdots) = \sum_{s \geqslant 0} \eta^s \mathscr{L}_s(\eta \partial, x)\, p(x, t \mid \cdots) \tag{4}$$

$$\mathscr{L}_s(\eta \partial, x) = \sum_{n \geqslant 1} \frac{(-\eta \partial)^n}{n!} a_n^{(s)}(x), \qquad s \geqslant 0 \tag{5}$$

with $a_n^{(s)}(x) = \sum_r r^n \omega_s(x, r)$. The conditional probability density can be written as

$$p(x, t \mid x', t') = \langle x \mid U(t, t') \mid x' \rangle$$

with the operator $U(t, t')$ satisfying

$$i \frac{\partial}{\partial t} U(t, t') = \hat{H} U(t, t'), \qquad \hat{H} = \frac{i}{\eta} \sum_{s \geqslant 0} \eta^s \mathscr{L}_s(\eta \partial \to i\eta \hat{p}, q \to \hat{q}) \tag{6}$$

where the operator $\hat{H}$ acts on functions $f(q)$, $\hat{q} f(q) = q f(q)$, $\hat{p} f(q) = -i(\partial/\partial q) f(q)$, $[\hat{q}, \hat{p}] = i$, and in $\hat{H}$ the operators $\hat{p}$ are on the left of the operators $\hat{q}$. From (6) it is immediate to write the generating functional of correlation and response functions of the Markov process defined by (4) as[3]

$$\begin{aligned} \bar{Z}[j, j^*] &= \int_{\gamma(0)} \mathscr{D}Q\, \mathscr{D}\bar{P} \exp i \int_{t_0}^{T} dt [\bar{P}\dot{Q} - \tilde{H}(\bar{P}, Q) + j(t)\, Q(t) + j^*(t)\, \bar{P}(t)] \\ &\quad \times \delta(Q(t_0) - \alpha_0) \end{aligned} \tag{7}$$

Here we have taken deterministic initial conditions $Q(t_0) = \alpha_0$ for the Markov process and $\gamma(0)$ stands for prepoint discretization (we omit this symbol from now on since we shall only use this discretization here), which defines the functional integral as the limit when $N \to \infty$ of the multiple integral I_N

$$\begin{aligned} I_N = \int_{Q_0 = \alpha_0} \prod_{i=1}^{N+1} dQ_i \prod_{j=1}^{N+1} \frac{d\bar{P}_j}{2\pi} \\ \times \exp i\varepsilon \sum_{j=1}^{N+1} \left[\bar{P}_j \frac{\Delta Q_j}{\varepsilon} - \tilde{H}(\bar{P}_j, Q_{j-1}) + j(t_j)\, Q_j + j^*(t_j)\, \bar{P}_j \right] \end{aligned} \tag{8}$$

with $\Delta Q_j = Q_j - Q_{j-1}$, $t_j = t_0 + j\varepsilon$, $t_{N+1} = T$, and $\varepsilon = (T - t_0)/(N+1)$. It is simple to check that $\bar{Z}[0, 0] = 1$, as it must be, since

$\bar{Z}[0, 0] = \int dq_{N+1}\, p(q_{N+1}, T \mid \cdots) = 1$. The function $\tilde{H}(\bar{P}, Q)$ is obtained from $\hat{H}$ by replacing $\hat{p} \to \bar{P}$, $\hat{q} \to Q$. The correlation functions are given by $(t_0 < \tau_j < T)$

$$G_m(\tau_1,..., \tau_m) = \langle q(\tau_1) \cdots q(\tau_m) \rangle = \prod_{l=1}^{m} \frac{1}{i} \frac{\delta}{\delta j(\tau_l)} \bar{Z}[j, j^*] \Big|_{j=j^*=0} \tag{9}$$

In (7) we make the change of variables $P(t) = \eta \bar{P}(t)$, and putting $J(t) = \sqrt{\eta}\, j(t)$, $j^*(t) = \sqrt{\eta}\, J^*(t)$, and $\bar{Z}[j, j^*] = Z[J, J^*]$, we obtain

$$\begin{aligned} Z[J, J^*] \\ &= \int \mathscr{D}Q\, \mathscr{D}^n P \exp \frac{i}{\eta} \int_{t_0}^{T} dt [P\dot{Q} - H_1(P, Q) - \eta H_2(P, Q, \eta) \\ &\quad + \sqrt{\eta}\, JQ + \sqrt{\eta}\, J^* P] \cdot \delta(Q(t_0) - \alpha_0) \end{aligned} \tag{10}$$

The discretized measure here is

$$\prod_{i=1}^{N+1} dQ_i \prod_{j=1}^{N+1} \frac{dP_j}{2\pi\eta}$$

All the dependence on η is now explicit in (10) with $H_1 = H^{(0)}$, $H_2 = \sum_{s \geqslant 0} \eta^s H^{(s+1)}(P, Q)$, and

$$H^s(P, Q) = i \sum_{n \geqslant 1} \frac{(-iP)^n}{n!} a_n^{(s)}(Q), \qquad s \geqslant 0 \tag{11}$$

We make now in (10) the change of variables

$$Q(t) = \alpha(t) + \sqrt{\eta}\, q(t), \qquad P(t) = \beta(t) + \sqrt{\eta}\, p(t) \tag{12}$$

where $(Q = \alpha(t),\ P = \beta(t))$ are solutions of Hamilton's equations for H_1, which are $[a_n(q) = a_n^{(0)}(q)]$

$$\dot{Q} = \frac{\partial H_1}{\partial P} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} P^n a_{n+1}(Q) \tag{13a}$$

$$\dot{P} = -\frac{\partial H_1}{\partial Q} = -i \sum_{n=1}^{\infty} \frac{(-iP)^n}{n!} \frac{da_n(Q)}{dQ} \tag{13b}$$

We take the solution $\beta(t) = 0$ and $\alpha(t)$ satisfying (the dot stands for derivative with respect to time)

$$\dot{\alpha}(t) = a_1(\alpha(t)), \qquad \alpha(t_0) = \alpha_0 \tag{14}$$

which is the deterministic equation in the sense that (4) reduces when $\eta \to 0$ to the Liouville equation $\partial_t p = \partial_x(-a_1(x))\,p$. From (10) we obtain (primes denote derivatives)

$$
\begin{aligned}
Z[J, J^*] = {} & \exp\left[\frac{i}{\sqrt{\eta}}\int_{t_0}^{T} dt\, J(t)\,\alpha(t)\right] \\
& \times \int \mathscr{D}q\,\mathscr{D}p \exp i\int_{t_0}^{T} dt \Bigg\{ p[\dot{q} - a_1'(\alpha(t))\,q] \\
& + \frac{i}{2} a_2(\alpha(t))\,p^2 + Jq + J^*p \\
& - \sum_{n+m\geqslant 3} \frac{(\sqrt{\eta})^{n+m-2}}{n!\,m!} \left.\frac{\partial^{n+m}H_1}{\partial P^n\,\partial Q^m}\right|_{\substack{P=0\\ Q=\alpha(t)}} p^n q^m \\
& - H_2(\sqrt{\eta}\,p, \alpha(t) + \sqrt{\eta}\,q, \eta)\Bigg\} \cdot \delta(q(t_0))
\end{aligned}
\tag{15}
$$

with

$$
\mathscr{D}q\,\mathscr{D}p = \prod_{i=1}^{N+1} dq_i \prod_{j=1}^{N+1} \frac{dp_j}{2\pi}
$$

in the discrete version. Using the generalization of the formula $f(q)\exp(Jq) = f(\partial/\partial J)\exp(Jq)$, we can write

$$
Z[J, J^*] = \exp\left[\frac{i}{\sqrt{\eta}}\int_{t_0}^{T} dt\, J(t)\,\alpha(t)\right] \cdot K[p, q]\Big|_{\substack{p=(1/i)\delta/\delta J^*\\ q=(1/i)\delta/\delta J}} \cdot Z_0[J, J^*] \tag{16a}
$$

$$
\begin{aligned}
Z_0[J, J^*] = {} & \int \mathscr{D}q\,\mathscr{D}p \exp i\int_{t_0}^{T} dt \Bigg\{ p[\dot{q} - a_1'(\alpha(t))\,q] \\
& + \frac{i}{2} a_2(\alpha(t))\,p^2 + Jq + J^*p\Bigg\} \cdot \delta(q(t_0))
\end{aligned}
\tag{16b}
$$

$$
\begin{aligned}
K[p, q] = {} & \exp\Bigg(-i\int_{t_0}^{T} dt \Bigg\{ \sum_{n+m\geqslant 3} \frac{(\sqrt{\eta})^{n+m-2}}{n!\,m!} \left.\frac{\partial^{n+m}H_1(P, Q)}{\partial P^n\,\partial Q^m}\right|_{\substack{P=0\\ Q=\alpha(t)}} \\
& \times p(t)^n\,q(t)^m \\
& + i\sum_{s\geqslant 0} \eta^s \sum_{n=1}^{\infty} \frac{[-i\sqrt{\eta}\,p(t)]^n}{n!} a_n^{(s+1)}[\alpha(t) + \sqrt{\eta}\,q(t)]\Bigg\}\Bigg)
\end{aligned}
\tag{16c}
$$

The functional $Z_0[J, J^*]$ is independent of η and can be calculated since it is a Gaussian functional integral. One obtains (Appendix A)

$$Z_0[J, J^*] = \exp\left[-i \int_{t_0}^{T} dt'\, dt''\, J(t')\, D(t', t'')\, J^*(t'') - \tfrac{1}{2} \int_{t_0}^{T} dt'\, dt''\, J(t')\, \Delta(t', t'')\, J(t'') \right] \tag{17a}$$

$$D(t, t') = \theta(t - t') \exp \int_{t'}^{t} ds\, a_1'(\alpha(s)) \tag{17b}$$

$$\begin{aligned} \Delta(t, t') = {} & \theta(t - t') \int_{t_0}^{t'} d\tau \left[\exp \int_{\tau}^{t'} ds\, a_1'(\alpha(s)) \right] \\ & \times a_2(\alpha(\tau)) \left[\exp \int_{\tau}^{t} ds\, a_1'(\alpha(s)) \right] \\ & + \theta(t' - t) \int_{t_0}^{t} d\tau \left[\exp \int_{\tau}^{t} ds\, a_1'(\alpha(s)) \right] \\ & \times a_2(\alpha(\tau)) \left[\exp \int_{\tau}^{t'} ds\, a_1'(\alpha(s)) \right] \end{aligned} \tag{17c}$$

where the step function $\theta(t) = 1$, $t > 0$, and $\theta(t) = 0$, $t < 0$. From (9) we obtain now that due to the change of sources the correlation functions are

$$G_m(\tau_1, ..., \tau_m) = \prod_{l=1}^{m} \frac{\sqrt{\eta}}{i} \frac{\delta}{\delta J(\tau_l)} Z[J, J^*] \Big|_{J = J^* = 0} \tag{18}$$

It is simple to check from (16) and (17) that (18) gives power series in η for the correlation functions. At this point the prepoint discretization which we are using is necessary to define completely the formal series, since from (17) we can see that $D(t + \varepsilon, t) \neq D(t, t + \varepsilon) = 0$, $\varepsilon \to +0$, while $\Delta(t, t')$ is a symmetric function of (t, t') and $\Delta(t + \varepsilon, t) = \Delta(t, t + \varepsilon)$, $\varepsilon \to +0$. We can write (16a) as

$$Z[J, J^*] = \exp\left[\frac{i}{\sqrt{\eta}} \int dt\, J(t)\, \alpha(t) \right] \hat{Z}[J, J^*], \quad \hat{Z}[0, 0] = 1$$

with

$$\hat{Z}[J, J^*] = K\left[\frac{1}{i} \frac{\delta}{\delta J^*}, \frac{1}{i} \frac{\delta}{\delta J} \right] Z_0[J, J^*] \tag{19}$$

In the right-hand side of (19) one has terms of the form

$$\frac{\delta}{\delta J(t)}\frac{\delta}{\delta J^*(t)}Z_0[J, J^*] = -iD(t, t) = -i\theta(0)$$

which are undefined, but since we are in the prepoint discretization these terms are

$$\lim_{\varepsilon\to+0}\frac{\delta}{\delta J(t)}\frac{\delta}{\delta J^*(t+\varepsilon)}Z_0[J, J^*] = -i\theta(-\varepsilon) = 0$$

since $p(t)\,q(t)$ is discretized as $p_j q_{j-1}$.[3] No problem arises with $\Delta(t, t)$ since this function is well defined at equal times. In summary, the prescription is that one must take $D(t, t) = 0$ in the calculation of $\hat{Z}[J, J^*]$. As an example of the calculation of (18) we can verify that when $\mathscr{L}_s = 0$, $s \geqslant 1$, the mean value $G_1(t)$ and the cumulant

$$\langle\langle x(t)\,x(t')\rangle\rangle = G_2(t, t') - G_1(t)\,G_1(t')$$

are given by

$$G_1(t) = \langle x(t)\rangle = \alpha(t) + \frac{\eta}{2}\int_{t_0}^{t} ds\, a_1''(\alpha(s))\,D(t, s)\,\Delta(s, s) + O(\eta^2) \tag{20a}$$

$$\langle\langle x(t)\,x(t')\rangle\rangle = \eta\Delta(t, t') + O(\eta^2) \tag{20b}$$

where the primes denote derivatives with respect to the argument. Formulas (16)–(18) give then in closed form the Ω-expansion of van Kampen.

We consider now the calculation of correlation functions in the stationary case. The Ω-expansion is local in the sense that one will obtain a system of correlation functions for each attractor of the deterministic equation $\dot{\alpha}(t) = a_1(\alpha(t))$. Let us suppose that $a_1(x)$ is as in Fig. 1. We see that $x = \mu$ and $x = \bar{\mu}$ are local attractors of the dynamical system

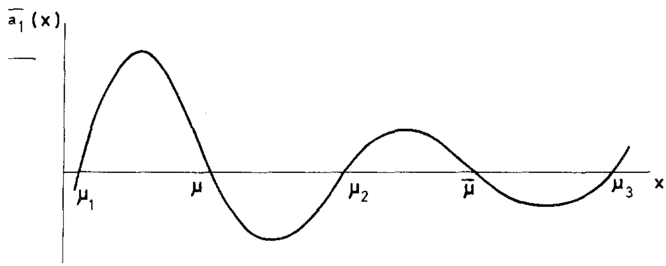


Fig. 1. Example of a deterministic equation.

$\dot{x} = a_1(x(t))$. If we consider the solution $\alpha(t; \alpha_0, t_0) = \tilde{\alpha}(\tau, \alpha_0)$, $\tau = t - t_0$, and $\alpha(t = t_0; \alpha_0, t_0) = \tilde{\alpha}(0; \alpha_0) = \alpha_0$, we see that for $\mu_1 < \alpha_0 < \mu_2$ (the basin of attraction of the equilibrium point μ) one has $\alpha(t; \alpha_0, t_0 \to -\infty) = \tilde{\alpha}(\infty, \alpha_0) = \mu$. The Ω-expansion is completely determined by this solution $\alpha(t)$ and the stationary solution is obtained by taking the limit $t_0 \to -\infty$ in (16) and (17) [we can of course put $T = +\infty$ in all our formulas, since (18) is independent of T for $\tau_i < T$]. The stationary values of the generating functionals are now

$$Z^{\text{stat}}[J, J^*] = \exp\left[\frac{i}{\sqrt{\eta}} \int_{-\infty}^{\infty} dt\, J(t)\, \mu\right] \times K^{\text{stat}}\left[\frac{1}{i}\frac{\delta}{\delta J^*}, \frac{1}{i}\frac{\delta}{\delta J}\right] \cdot Z_0^{\text{stat}}[J, J^*] \tag{21a}$$

$$Z_0^{\text{stat}}[J, J^*] = \exp\left[-i \int_{-\infty}^{\infty} dt'\, dt''\, J(t')\, D^{\text{stat}}(t' - t'')\, J^*(t'') - \frac{1}{2}\int_{-\infty}^{\infty} dt'\, dt''\, J(t')\, \Delta^{\text{stat}}(t' - t'')\, J(t'')\right] \tag{21b}$$

$$D^{\text{stat}}(t) = \theta(t)\, e^{a_1'(\mu) t}, \qquad \Delta^{\text{stat}}(t) = -\frac{a_2(\mu)}{2a_1'(\mu)}\, e^{a_1'(\mu)|t|} \tag{21c}$$

and all integrals are well defined since $a_1'(\mu) < 0$ because μ is a local attractor. From (20) we obtain now for the time-independent mean value in the stationary state

$$G_1 = \langle x(\tau) \rangle^{\text{stat}} = \mu + \eta\, \frac{a_1''(\mu)\, a_2(\mu)}{2a_1'(\mu)^2} + O(\eta^2) \tag{22}$$

We can repeat all our calculations with a solution $\alpha_1(t)$, $\dot{\alpha}_1 = a_1(\alpha_1(t))$, $\alpha_1(t_0) = \bar{\alpha}_0$, and $\mu_2 < \bar{\alpha}_0 < \mu_3$; then $\alpha_1(t) \to \bar{\mu}$ when $t_0 \to -\infty$ and the correlation functions will represent fluctuations around the deterministic solution $\alpha_1(t)$ instead of fluctuations around $\alpha(t)$ as before. If there is only one attractor for the dynamical system $\dot{x} = a_1(x)$, then the results are global. In the case of coexistence of attractors the fluctuations that we determine around each of them have a meaning insofar as the escape time from the attractor is much bigger than the times in which we are interested. If this is not the case, we are in a critical region and we have to modify the expansion.[7] The escape times have to be determined from the global stationary probability $p_{\text{st}}(q)$ of the process and its dominant behavior is $\exp(b/\eta)$, $b > 0$, and the expansion is then meaningful asymptotically for small $\eta = \Omega^{-1}$. The expansion is of course meaningless around an unstable equilibrium point like μ_1 in Fig. 1 since $a_1'(\mu_1) > 0$ and the integrals in (21) diverge.

If the original transition probability W depends explicitly on time, everything goes through in the same way, but now $H_1(P, Q, t)$ and $H_2(P, Q, t)$ in (10) will be time dependent and this dependence will appear in the rest of the calculation. The existence of a stationary state has to be examined case by case in this situation.

In practical calculations it is convenient to reformulate our method. The initial equation (4) can be written (all derivatives are on the left of functions of x)

$$i\eta \frac{\partial}{\partial t} p(x, t \mid x', t') = \left[H_1\left(-i\eta \frac{\partial}{\partial x}, x \right) + \eta H_2\left(-i\eta \frac{\partial}{\partial x}, x \right) \right] p(x, t \mid x', t') \tag{23}$$

where H_1 and H_2 were defined after Eq. (10) and are given by

$$H_1(P, Q) = i\mathscr{L}_0(iP, Q) \tag{24a}$$

$$H_2(P, Q) = i \sum_{s \geq 1} \eta^{s-1} \mathscr{L}_s(iP, Q) \tag{24b}$$

We make in (23) the time-dependent change of variables $x = \alpha(t) + \sqrt{\eta}\, q$ and putting $\bar{p}(q, t \mid q', t') = p(x, t \mid x', t')$, we obtain

$$\begin{aligned} i\eta \frac{\partial}{\partial t} \bar{p}(q, t \mid q', t') &= \left[H_1\left(-i\sqrt{\eta} \frac{\partial}{\partial q}, \alpha(t) + \sqrt{\eta}\, q \right) \right. \\ &\quad + \eta H_2\left(-i\sqrt{\eta} \frac{\partial}{\partial q}, \alpha(t) + \sqrt{\eta}\, q \right) \\ &\quad \left. + i\sqrt{\eta}\, \dot{\alpha}(t) \frac{\partial}{\partial q} \right] \bar{p}(q, t \mid q', t') \end{aligned} \tag{25}$$

From (25) we can write the generating functional $\tilde{Z}[J, J^*]$ of correlation functions[3,6] of the process $q(t) = \eta^{-1/2}[x(t) - \alpha(t)]$ with initial condition $q(t_0) = 0$ as

$$\tilde{Z}[J, J^*] = \int \mathscr{D}q\, \mathscr{D}p \exp i \int_{t_0}^{T} dt [p\dot{q} - \bar{H}(p, q, t) + Jq + J^* p] \cdot \delta(q(t_0)) \tag{26a}$$

$$\begin{aligned} \eta \bar{H}(p, q, t) &= H_1(\sqrt{\eta}\, p, \alpha(t) + \sqrt{\eta}\, q) + \eta H_2(\sqrt{\eta}\, p, \alpha(t) + \sqrt{\eta}\, q) \\ &\quad - \sqrt{\eta}\, \dot{\alpha}(t)\, p \end{aligned} \tag{26b}$$

We expand now $\bar{H}(P, Q, t)$ around $(0, \alpha(t))$ $[f(P, Q)|_0$ means that we evaluate at $P=0$, $Q=\alpha(t)]$

$$\bar{H}=\frac{1}{2}\frac{\partial^2 H_1(P, Q)}{\partial P^2}\bigg|_0 p^2+\frac{\partial^2 H_1(P, Q)}{\partial P\,\partial Q}\bigg|_0 pq+M(p, q, t) \qquad (27a)$$

$$M(p, q, t)=\sum_{n+m\geqslant 3}\frac{(\sqrt{\eta})^{n+m-2}}{n!\,m!}\frac{\partial^{n+m}H_1(P, Q)}{\partial P^n\,\partial Q^m}\bigg|_0 p^n q^m + H_2(\sqrt{\eta}\,p, \alpha(t)+\sqrt{n}\,q) \qquad (27b)$$

and then we can write $\tilde{Z}$ in the form

$$\tilde{Z}[J, J^*]=K\left[\frac{1}{i}\frac{\delta}{\delta J^*}, \frac{1}{i}\frac{\delta}{\delta J}\right] Z_0[J, J^*] \qquad (28)$$

where

$$K[p, q]=\exp\left[-i\int_{t_0}^{T} dt\, M(p, q, t)\right]$$

and Z_0 are given by (16). We see then from (19) that we have $\tilde{Z}[J, J^*]=\hat{Z}[J, J^*]$, which gives the interpretation of $\hat{Z}$ as the generating functional of the process $q(t)$ in the sense that the correlation functions $\tilde{G}_m(\tau_1\cdots\tau_m)$ of this process have the value

$$\tilde{G}_m(\tau_1\cdots\tau_m)=\langle q(\tau_1)\cdots q(\tau_m)\rangle=\prod_{l=1}^{m}\frac{1}{i}\frac{\delta}{\delta J(\tau_l)}\hat{Z}[J, J^*]\bigg|_{J=J^*=0} \qquad (29)$$

Correlation functions $G_m(\tau_1\cdots\tau_m)=\langle x(\tau_1)\cdots x(\tau_m)\rangle$ of the initial process $x(t)$ are related to $\tilde{G}_m$ in an obvious way through $x(t)=\alpha(t)+\sqrt{\eta}\,q(t)$ and the most convenient method is to calculate using (29).

3. APPLICATIONS OF THE FORMALISM

We shall explain in this section the most convenient way to make practical calculations. The basic formula is (29), and using (19), one has

$$\tilde{G}_m(\tau_1\cdots\tau_m)=\prod_{l=1}^{m}\frac{1}{i}\frac{\delta}{\delta J(\tau_l)}K\left[\frac{1}{i}\frac{\delta}{\delta J^*}, \frac{1}{i}\frac{\delta}{\delta J}\right] Z_0[J, J^*]\bigg|_{J=J^*=0} \qquad (30)$$

We use here the generalization of the formula

$$F\left(\frac{1}{a}\frac{\partial}{\partial J}\right) Z_0(J)\bigg|_{J=0}=Z_0\left(\frac{1}{a}\frac{\partial}{\partial q}\right) F(q)\bigg|_{q=0},$$

valid for any functions $F(\cdot)$ and $Z_0(\cdot)$. From (30) we obtain

$$\tilde{G}_m(\tau_1 \cdots \tau_m) = Z_0\left[\frac{1}{i}\frac{\delta}{\delta q}, \frac{1}{i}\frac{\delta}{\delta p}\right] q(\tau_1)\cdots q(\tau_m)\, K[p, q]\bigg|_{p=q=0} \tag{31}$$

We put $M = iH_I(p, q, t)$, $H_I = \sqrt{\eta}\sum_{n\geq 0}(\sqrt{\eta})^n H_I^{(n)}$, since M is a formal power series in $\sqrt{\eta}$. Then

$$\begin{aligned} K[p, q] &= \exp\int_{t_0}^{T} dt\, H_I(p, q, t) \\ &= 1 + \sqrt{\eta}\int_{t_0}^{T} dt\, H_I^{(0)}(t) + \eta\left[\int_{t_0}^{T} dt\, H_I^{(1)}(t)\right. \\ &\quad \left. + \tfrac{1}{2}\int_{t_0}^{T} dt_1\, dt_2\, H_I^{(0)}(t_1)\, H_I^{(0)}(t_2)\right] \\ &\quad + O(\sqrt{\eta}^3) \end{aligned} \tag{32}$$

and each $\tilde{G}_m$ will be a formal series in powers of $\sqrt{\eta}$. We remark that (16c) shows that the term of order $(\sqrt{\eta})^{2n+1}$ in $K[p, q]$ [respectively, of order $(\sqrt{\eta})^{2n}$] will contain a product $p(\tau_1)\cdots p(\tau_l)\, q(\tau'_1)\cdots q(\tau'_k)$ with $(l+k)$ odd (resp. even). This implies that the expansion of $\tilde{G}_m$ for m even (resp. m odd) will have only even powers $(\sqrt{\eta})^{2n} = \eta^n$ [resp. only odd powers $(\sqrt{\eta})^{2n+1} = \sqrt{\eta}\,\eta^n$] due to the fact that $Z_0[J, J^*]$ is the exponential of a quadratic form in (J, J^*). Putting $K = 1 + \sum_{n\geq 1}(\sqrt{\eta})^n K_n$, one has that each K_n will be a multiple integral where the integrand is a product of given functions of the time with products $(p(\tau_1)\, p(\tau_2)\cdots q(\tau'_1)\, q(\tau'_2)\cdots)$. Then each term in (31) will be of the form of a multiple integral and in the integrand one will have

$$\begin{aligned} Z_0\left[\frac{1}{i}\frac{\delta}{\delta q}, \frac{1}{i}\frac{\delta}{\delta p}\right](p(\tau_1)\, p(\tau_2)\cdots q(\tau'_1)\, q(\tau'_2)\cdots)\bigg|_{p=q=0} \\ = \{p(\tau_1)\, p(\tau_2)\cdots q(\tau'_1)\, q(\tau'_2)\cdots\} \end{aligned} \tag{33}$$

where (33) defines the notation $\{\cdots\}$. Due to the form of Z_0 one has

$$\{p(\tau_1)\, p(\tau_2)\} = 0, \quad \{q(\tau_1)\, p(\tau_2)\} = iD(\tau_1, \tau_2), \quad \{q(\tau_1)\, q(\tau_2)\} = \Delta(\tau_1, \tau_2) \tag{34}$$

and $\{q(\tau)\, p(\tau)\} = 0$ due to the observation after Eq. (19). In order to give the value of the general term (33), we put $z_1(\tau) = p(\tau)$ and $z_2(\tau) = q(\tau)$

and we can easily see [using the short-hand notation $z_i = z(\tau_i)$] that $\{z_1 \cdots z_{2n+1}\} = 0$ and

$$\{z_1 z_2 \cdots z_{2n}\} = \overline{z_1 z_2}\,\overline{z_3 z_4} \cdots \overline{z_{2n-1} z_{2n}} + \text{(all possible groups of pairs)} \tag{35}$$

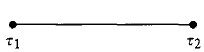
where

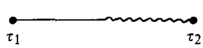
$$\overline{z_\mu(\tau) z_\nu(\tau')} = \{z_\mu(\tau)\, z_\nu(\tau')\} = S_{\mu\nu}(\tau, \tau') \tag{36}$$

is called a contraction and from (34) one has $S_{11}(\tau, \tau') = 0$, $S_{12}(\tau, \tau') = iD(\tau', \tau)$, $S_{21}(\tau, \tau') = iD(\tau, \tau')$, and $S_{22}(\tau, \tau') = \Delta(\tau, \tau')$. The number of terms in the right-hand side of (35) is $(2n-1)!! = (2n-1)(2n-3) \cdots 3 \cdot 1$. For example, in the case $n = 2$ one has

$$\{z_1 z_2 z_3 z_4\} = \overline{z_1 z_2}\,\overline{z_3 z_4} + \overline{z_1 z_3}\,\overline{z_2 z_4} + \overline{z_1 z_4}\,\overline{z_2 z_3} \tag{37}$$

Formula (35) is called the Wick theorem (see, for example, ref. 5). Each term in the expansion of the right-hand side of (30) can be represented by a graph which is determined by the contractions in (35). We put

$$\overline{q(\tau_1) q(\tau_2)} = \Delta(\tau_1, \tau_2) = $$  (38a)

$$\overline{q(\tau_1) p(\tau_2)} = iD(\tau_1, \tau_2) = $$  (38b)

As an example we consider $\{q(t')\, q(t'')\, p(\tau)^2\, q(\tau)^2\}$. One of the terms here is

$$\overline{q(t') q(t'') p(\tau)}\, p(\tau) \overline{q(\tau) q(\tau)} = \overline{q(t') p(\tau)}\; \overline{q(t'') p(\tau)}\; \overline{q(\tau) q(\tau)}$$
$$= iD(t', \tau)\, iD(t'', \tau)\, \Delta(\tau, \tau)$$

which correspond to the graph in Fig. 2.

We shall apply these simple rules to calculate up to $O(\eta^2)$ the cumulant for a model of a semiconductor. Introducing as the intensive variable the density of excited electrons $x = X/\Omega$, where $X = 1, 2, \ldots$ is the number of excited electrons, we find that the master equation takes here the form (4) (see ref. 2, Chapter VI, for a discussion of the model) with $\mathscr{L}_s = 0$, $s \geq 1$, and

$$a_n^{(0)}(x) \equiv a_n(x) = b + (-1)^n a x^2 \tag{39}$$

where a and b are positive constants. The deterministic (macroscopic) equation $\dot{\alpha}(t) = a_1(\alpha(t))$ has the solution

$$\alpha(t) = \left(\frac{b}{a}\right)^{1/2} \frac{\alpha_0 + (b/a)^{1/2} \tanh(ab)^{1/2}\,(t - t_0)}{(b/a)^{1/2} + \alpha_0 \tanh(ab)^{1/2}\,(t - t_0)}, \qquad \alpha(t_0) = \alpha_0 \tag{40}$$

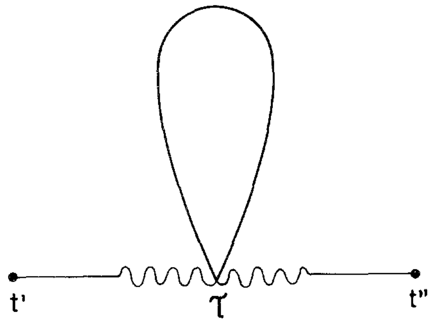


Fig. 2. Graph contributing to the correlation function.

The initial master equation has natural boundary conditions and for $\alpha_0 > 0$ one has $\alpha(t) = \tilde{\alpha}(\tau = t - t_0) \to \mu = (b/a)^{1/2}$, $\tau \to \infty$, which is a global attractor. One has $a_1'(\mu) = -2(ab)^{1/2} < 0$. The functional Z_0 is given by (17a) with

$$D(t, t') = \theta(t - t') \left(\frac{\cosh \rho'}{\cosh \rho}\right)^2 \left(\frac{(b/a)^{1/2} + \alpha_0 \tanh \rho'}{\alpha_0 + (b/a)^{1/2} \tanh \rho}\right) \tag{41a}$$

$$\Delta(t, t') = D(t, t')\, F(t') + D(t', t)\, F(t) \tag{41b}$$

$$\begin{aligned} F(t) = \frac{1}{2}\left(\frac{b}{a}\right)^{1/2} \Bigg[&\frac{\alpha_0 + (b/a)^{1/2} \tanh \rho}{(b/a)^{1/2} + \alpha_0 \tanh \rho} \\ &+ \frac{1}{2}\left(\frac{b}{a} - \alpha_0^2\right) \frac{1}{(\cosh \rho)^2\, [(b/a)^{1/2} + \alpha_0 \tanh \rho]^2} \frac{\alpha_0 + (b/a) \tanh \rho}{(b/a)^{1/2} + \alpha_0 \tanh \rho} \\ &+ \frac{1}{2}\left(\frac{b}{a}\right)^{1/2} \alpha_0 \left(3\frac{b}{a} - \alpha_0^2\right) \frac{1}{(\cosh \rho)^4\, [(b/a)^{1/2} + \alpha_0 \tanh \rho]^4} \\ &+ \frac{1}{2}\left(\frac{b}{a} - \alpha_0^2\right)^2 \frac{\rho}{(\cosh \rho)^4\, [(b/a)^{1/2} + \alpha_0 \tanh \rho]^4} \Bigg] \end{aligned} \tag{41c}$$

where $\rho = (ab)^{1/2}\,(t - t_0)$ and $\rho' = (ab)^{1/2}\,(t' - t_0)$. We put $x(t) = \alpha(t) + \sqrt{\eta}\, q(t)$ and the two-point cumulant will be

$$\langle\!\langle x(t')\, x(t'') \rangle\!\rangle = \langle x(t')\, x(t'') \rangle - \langle x(t') \rangle \langle x(t'') \rangle = \eta \langle\!\langle q(t')\, q(t'') \rangle\!\rangle \tag{42}$$

We have then to calculate

$$\begin{aligned} \langle\!\langle q(t')\, q(t'') \rangle\!\rangle &= \langle q(t')\, q(t'') \rangle - \langle q(t') \rangle \langle q(t'') \rangle \\ &= \tilde{G}_2(t', t'') - \tilde{G}_1(t')\, \tilde{G}_1(t'') \end{aligned}$$

using (31). An important point here is that it is not necessary to make explicitly the subtraction $(\tilde{G}_2 - \tilde{G}_1 \tilde{G}_1)$, since it can be easily checked that if in the expansion (31) for $\tilde{G}_2$ we keep only the connected graphs (those which are not formed by two or more separated parts, such as the graph in Fig. 2), we obtain directly $\langle\langle q(t')\, q(t'') \rangle\rangle$. This is true for any cumulant of the process $q(t)$. We want to calculate in our model up to $O(\eta^2)$. We put

$$\langle\langle q(t')\, q(t'') \rangle\rangle = S_0(t', t'') + \eta S_1(t', t'') + O(\eta^2) \tag{43}$$

From (31) we obtain $S_0(t', t'') = q(t')\, q(t'') = \Delta(t', t'')$. On the other hand, we see from (32) that we need $H_I^{(0)}$ and $H_I^{(1)}$,

$$H_I^{(0)} = \frac{i}{3!} a_3(\alpha(t))\, p^3 - \frac{1}{2} a_{2,1}(\alpha(t))\, p^2 q - \frac{i}{2} a_{1,2}(\alpha(t))\, pq^2 \tag{44a}$$

$$H_I^{(1)} = \frac{1}{4!} a_4(\alpha(t))\, p^4 + \frac{i}{3!} a_{3,1}(\alpha(t))\, p^3 q - \frac{1}{4} a_{2,2}(\alpha(t))\, p^2 q^2 \tag{44b}$$

with $a_n(x)$ given by (39) and $a_{n,m}(x) = d^m a_n(x)/dx^m$. It is simple to see that

$$\{q(t')\, q(t'')\, H_I^{(0)}(t)\} = 0$$

and then from (30) and (32) we obtain

$$S_1(t', t'') = S_1^{(1)}(t', t'') + S_1^{(2)}(t', t'') \tag{45a}$$

$$S_1^{(1)} = \int_{t_0}^{T} d\tau\, \{q(t')\, q(t'')\, H_I^{(1)}(\tau)\} \tag{45b}$$

$$S_1^{(2)}(t', t'') = \tfrac{1}{2} \int_{t_0}^{T} d\tau_1\, d\tau_2\, \{q(t')\, q(t'')\, H_I^{(0)}(\tau_1)\, H_I^{(0)}(\tau_2)\} \tag{45c}$$

where only connected graphs have to be considered in (45b) and (45c). One has

$$a_{n,1}(x) = (-1)^n\, 2ax, \qquad a_{n,2}(x) = (-1)^n\, 2a \tag{46}$$

and in $S_1^{(1)}$ only one term arises with the value

$$S_1^{(1)}(t', t'') = a \int_{t_0}^{\min(t', t'')} d\tau\, D(t', \tau)\, D(t'', \tau)\, \Delta(\tau, \tau) \tag{47}$$

which corresponds to the graph of Fig. 2. For $S_1^{(2)}$ we obtain

$$S_1^{(2)}(t', t'') = 2a^2 \left[\int d\tau_1\, d\tau_2\, D(t', \tau_1)\, D(\tau_1, \tau_2)\, \Delta(\tau_1, t'')\, \Delta(\tau_2, \tau_2) \right.$$
$$+ 2 \int d\tau_1\, d\tau_2\, D(t', \tau_1)\, D(\tau_1, \tau_2)\, \Delta(\tau_1, \tau_2)\, \Delta(t'', \tau_2)$$
$$\left. + \tfrac{1}{2} \int d\tau_1\, d\tau_2\, D(t', \tau_1)\, D(t'', \tau_2)\, \Delta(\tau_1, \tau_2)^2 \right]$$
$$- a \int d\tau_1\, d\tau_2\, a_3(\alpha(\tau_1))\, D(t', \tau_1)\, D(\tau_1, \tau_2)^2\, D(t'', \tau_2)$$
$$- a \left[\tfrac{1}{2} \int d\tau_1\, d\tau_2\, a_{2,1}(\alpha(\tau_1))\, D(t', \tau_1)\, D(t'', \tau_1)\, D(\tau_1, \tau_2)\, \Delta(\tau_2, \tau_2) \right.$$
$$+ 2 \int d\tau_1\, d\tau_2\, a_{2,1}(\alpha(\tau_1))\, D(t', \tau_1)\, D(t'', \tau_2)\, D(\tau_2, \tau_1)\, \Delta(\tau_1, \tau_2)$$
$$\left. + \int d\tau_1\, d\tau_2\, a_{2,1}(\alpha(\tau_1))\, D(t'', \tau_2)\, \Delta(t', \tau_1)\, D(\tau_2, \tau_1)^2 \right]$$
$$+ (t' \leftrightarrow t'') \tag{48}$$

where $(t' \leftrightarrow t'')$ means that we have to add all the preceding terms after interchanging t' and t''. In (48) all integrals are in $[t_0, T]$, but the result is of course independent of $T > (t', t'')$, as can be checked by recalling that $D(t, t')$ is proportional to $\theta(t - t')$. From (43) and (48) we obtain the cumulant in the stationary state, taking the limit $t_0 \to -\infty$. In this limit $\alpha(t) \to \mu = (b/a)^{1/2}$, $D \to D^{\mathrm{st}}$, and $\Delta \to \Delta^{\mathrm{st}}$, with

$$D^{\mathrm{st}}(t', t'') = \theta(t - t') \exp[-|a_{1,1}(\mu)|\, (t - t')] \tag{49a}$$

$$\Delta^{\mathrm{st}}(t, t') = \frac{a_2(\mu)}{2\, |a_{1,1}(\mu)|} \exp[-|a_{1,1}(\mu)|\, |(t - t')|] \tag{49b}$$

and $a_{1,1}(\mu) = -2(ab)^{1/2}$. We obtain

$$\langle\langle q(t)\, q(t') \rangle\rangle^{\mathrm{st}}$$
$$= \frac{1}{2} \left(\frac{b}{a} \right)^{1/2} \exp[-2(ab)^{1/2}\, |t - t'|]$$
$$+ \eta \left\{ -\frac{1}{8} \exp[-2(ab)^{1/2}\, |t - t'|] \right.$$
$$+ \frac{1}{4} (ab)^{1/2}\, |t - t'| \exp[-2(ab)^{1/2}\, |t - t'|]$$
$$\left. + \frac{1}{8} \exp[-4(ab)^{1/2}\, |t - t'|] \right\} \tag{50}$$

APPENDIX A

We calculate here $Z_0[J, J^*]$ given by (16b). The method is the same as used in ref. 3, Chapter IX, for the case in which the coefficients of pq and p^2 are time independent. One has that $Z_0[J, J^*] = \int \mathscr{D}q\, \mathscr{D}p\, B[p, q]$ and the integration-by-parts lemma implies

$$\int \mathscr{D}q\, \mathscr{D}p \frac{\delta}{\delta q(t)} B[p, q] = \int \mathscr{D}q\, \mathscr{D}p \frac{\delta}{\delta p(t)} B[p, q] = 0 \tag{A1}$$

This gives the following two equations:

$$\left[\frac{\partial}{\partial t} + a_1'(\alpha(t))\right] \frac{1}{i} \frac{\delta Z_0}{\delta J^*(t)} = J(t)\, Z_0[J, J^*] \tag{A2}$$

$$\left[\frac{\partial}{\partial t} - a_1'(\alpha(t))\right] \frac{1}{i} \frac{\delta Z_0}{\delta J(t)} = -a_2(\alpha(t)) \frac{\delta Z_0}{\delta J^*(t)} - J^*(t)\, Z_0[J, J^*] \tag{A3}$$

which have to be solved with the boundary conditions

$$\left.\frac{1}{i} \frac{\delta Z_0}{\delta J(t)}\right|_{t=t_0} = 0, \qquad \left.\frac{1}{i} \frac{\delta Z_0}{\delta J^*(t)}\right|_{t=T} = 0, \qquad Z_0[0, 0] = 1 \tag{A4}$$

The solution of (A2) is

$$\frac{1}{i} \frac{\delta Z_0}{\delta J^*(t)} = -\int_{t_0}^{T} dt'\, J(t')\, D(t', t) \cdot Z_0[J, J^*] \tag{A5}$$

with $D(t', t)$ given by (17b). We replace this value in (A3) and solve this equation to obtain

$$\begin{aligned} \frac{1}{i} \frac{\delta Z_0}{\delta J(t)} = &-\int_{t_0}^{T} dt'\, D(t, t')\, J^*(t, t') \cdot Z_0[J, J^*] \\ &+ i \int_{t_0}^{T} dt'\, \Delta(t, t')\, J(t') \cdot Z_0[J, J^*] \end{aligned} \tag{A6}$$

with $\Delta(t, t')$ given by (17c). Equations (A5) and (A6) together with $Z_0[0, 0] = 1$ give then the form (17a).

APPENDIX B

We consider now the case in which the original variable is a vector. Equation (3) will be of the form

$$\eta \frac{\partial}{\partial t} p(\mathbf{x}, t \mid \cdots) = \sum_{s \geqslant 0} \eta^s \sum_{r} (e^{-\eta r_\mu \partial_\mu} - 1)\, \omega_s(\mathbf{x}, \mathbf{r})\, p(\mathbf{x}, t \mid \cdots) \tag{B1}$$

with $\mathbf{x}=(x_1,..., x_M)$, $\boldsymbol{\partial}=(\partial_1,..., \partial_M)$, $\partial_\mu=\partial/\partial x_\mu$, and $\mathbf{r}=(r_1,..., r_M)$, and summation over repeated indices must be from now on understood $(r_\mu \partial_\mu = \sum_{\mu=1}^{M} r_\mu \partial_\mu)$. Equation (4) becomes

$$\eta \frac{\partial}{\partial t} p(\mathbf{x}, t \mid \cdots) = \sum_{s \geqslant 0} \eta^s \mathscr{L}_s(\eta \boldsymbol{\partial}, \mathbf{x})\, p(\mathbf{x}, t \mid \cdots) \tag{B2}$$

$$\mathscr{L}_s(\eta \boldsymbol{\partial}, \mathbf{x}) = \sum_{n \geqslant 1} \frac{(-\eta)^n}{n!} \partial_{\mu_1} \cdots \partial_{\mu_n} a^{(s)}_{\mu_1 \cdots \mu_n}(\mathbf{x}) \tag{B3}$$

$$a^{(s)}_{u_1 \cdots \mu_n}(\mathbf{x}) = \sum_{\mathbf{r}} r_{\mu_1} \cdots r_{\mu_n} \omega_s(\mathbf{x}, \mathbf{r}) \tag{B4}$$

The generating functional of correlation functions for deterministic initial conditions $\mathbf{x}(t_0)=\boldsymbol{\alpha}^0$ will be

$$\begin{aligned} \bar{Z}[\mathbf{j}, \mathbf{j}^*] = \int_{\gamma(0)} & \mathscr{D}\mathbf{Q}\, \mathscr{D}\bar{\mathbf{P}} \\ & \times \exp i \int_{t_0}^{T} dt\, [\bar{P}_\mu \dot{Q}_\mu - \tilde{H}(\bar{\mathbf{P}}, \mathbf{Q}) + j_\mu Q_\mu + j^*_\mu \bar{P}_\mu] \\ & \times \delta(\mathbf{Q}(t_0) - \boldsymbol{\alpha}^0) \end{aligned} \tag{B5}$$

where

$$\tilde{H}(\bar{\mathbf{P}}, \mathbf{Q}) = \frac{i}{\eta} \sum_{s \geqslant 0} \eta^s \mathscr{L}_s(\eta \boldsymbol{\partial} \to i\eta \bar{\mathbf{P}}, \mathbf{x} \to \mathbf{Q}) \tag{B6}$$

and $\gamma(0)$ (which we omit from now on) stands for prepoint discretization, which defines $\bar{Z}$ in (B5) as $\lim I_N$, $N \to \infty$, with

$$\begin{aligned} I_N = \int_{Q_{\mu,0}=\alpha^0_\mu} \prod_{\substack{i=1 \\ u=1,...,M}}^{N+1} dQ_{\mu,i} \prod_{\substack{j=1 \\ v=1,...,M}}^{N+1} \frac{d\bar{P}_{v,j}}{2\pi} \exp i\varepsilon \sum_{j=1}^{N+1} \Bigg[\bar{P}_{\mu,j} \frac{\Delta Q_{\mu,j}}{\varepsilon} \\ - \tilde{H}(\bar{\mathbf{P}}_j, \mathbf{Q}_{j-1}) + j_\mu(t_j)\, Q_{\mu,j} + j^*_\mu(t_j)\, \bar{P}_{\mu,j} \Bigg] \end{aligned} \tag{B7}$$

with the obvious generalization of the definitions in the text which we shall use in this Appendix. Proceeding as in Section 2, we have that H_1 and H_2 in (10) are now

$$\begin{aligned} & H_1 = H^{(0)}, \qquad H_2 = \sum_{s \geqslant 0} \eta^s H^{(s+1)} \\ & H^{(s)}(\mathbf{P}, \mathbf{Q}) = i \sum_{n \geqslant 1} \frac{(-i)^n}{n!} P_{\mu_1} \cdots P_{\mu_n} a^{(s)}_{\mu_1 \cdots \mu_n}(\mathbf{Q}) \end{aligned} \tag{B8}$$

Putting $J_\mu = \sqrt{\eta}\, j_\mu$, $j^*_\mu = \sqrt{\eta}\, J^*_\mu$, and $Z[\mathbf{J}, \mathbf{J}^*] = \bar{Z}[\mathbf{j}, \mathbf{j}^*]$, we have

$$Z[\mathbf{J}, \mathbf{J}^*] = \exp\left[\frac{i}{\sqrt{\eta}} \int_{t_0}^{T} dt\, J_\mu(t)\, \alpha_\mu(t)\right] \cdot \hat{Z}[\mathbf{J}, \mathbf{J}^*]$$

with

$$\hat{Z}[\mathbf{J}, \mathbf{J}^*] = K\left[\frac{1}{i}\frac{\delta}{\delta \mathbf{J}^*}, \frac{1}{i}\frac{\delta}{\delta \mathbf{J}}\right] \cdot Z_0[\mathbf{J}, \mathbf{J}^*] \tag{B9}$$

$$Z_0[\mathbf{J}, \mathbf{J}^*] = \int \mathscr{D}\mathbf{q}\, \mathscr{D}\mathbf{p} \exp i \int_{t_0}^{T} dt \{ p_\mu [\dot{q}_\mu - \partial_\rho a_\mu(\boldsymbol{\alpha}(t))\, q_\rho] + \frac{i}{2} a_{\mu\nu}(\boldsymbol{\alpha}(t))\, p_\mu p_\nu + J_\mu q_\mu + J^*_\mu p_\mu \} \cdot \delta(\mathbf{q}(t_0)) \tag{B10}$$

$$K[\mathbf{p}, \mathbf{q}] = \exp\left\{ -i \int_{t_0}^{T} dt \left[\sum_{n+m \geqslant 3} \frac{(\sqrt{\eta})^{n+m-2}}{n!\, m!} \times \frac{\partial^{n+m} H_1(\mathbf{P}, \mathbf{Q})}{\partial P_{\mu_1} \cdots P_{\mu_n} \partial Q_{\nu_1} \cdots \partial Q_{\nu_m}} \bigg|_{\substack{\mathbf{P}=\mathbf{0} \\ \mathbf{Q}=\boldsymbol{\alpha}(t)}} q_{\mu_1} \cdots q_{\mu_m} p_{\nu_1} \cdots p_{\nu_n} + i \sum_{s \geqslant 0} \eta^s \sum_{n \geqslant 1} \frac{(-i\sqrt{\eta})^n}{n!} p_{\mu_1} \cdots p_{\mu_n} a^{(s+1)}_{\mu_1 \cdots \mu_n}(\boldsymbol{\alpha}(t) + \sqrt{\eta}\, \mathbf{q}(t)) \right] \right\} \tag{B11}$$

where $a_{\mu_1 \cdots \mu_n} \equiv a^{(0)}_{u_1 \cdots \mu_n}$ and $\boldsymbol{\alpha}(t)$ is the solution of the deterministic equations

$$\dot{\alpha}_\mu(t) = a_\mu(\boldsymbol{\alpha}(t)), \qquad \alpha_\mu(t_0) = \alpha^0_\mu \tag{B12}$$

Formulas (B9)–(B11) allow us to calculate the correlation functions

$$G_{\mu_1 \cdots \mu_m}(\tau_1, \ldots, \tau_m) = \langle x_{\mu_1}(\tau_1) \cdots x_{\mu_m}(\tau_m) \rangle = \prod_{l=1}^{m} \frac{\sqrt{\eta}}{i} \frac{\delta}{\delta J_{\mu_l}(\tau_l)} Z[\mathbf{J}, J^*] \bigg|_{\mathbf{J} = \mathbf{J}^* = 0} \tag{B13}$$

as formal series in powers of η, since Z_0 is independent of η. We calculate now Z_0 using the same method as in Appendix A. Putting $B_{\mu\rho}(\mathbf{q}) = \partial_\rho a_\mu(\mathbf{q})$, we have instead of (A2) and (A3) the equations

$$\left[\frac{\partial}{\partial t} \delta^\mu_\rho + B_{\mu\rho}(\boldsymbol{\alpha}(t))\right] \frac{1}{i} \frac{\delta Z_0}{\delta J^*_\rho(t)} = J_\mu(t)\, Z_0 \tag{B14}$$

$$\left[\frac{\partial}{\partial t} \delta^\mu_\rho - B_{\mu\rho}(\boldsymbol{\alpha}(t))\right] \frac{1}{i} \frac{\delta Z_0}{\delta J_\rho(t)} = -a_{\mu\nu} \frac{\delta Z_0}{\delta J^*_\nu(t)} - J^*_\mu(t)\, Z_0 \tag{B15}$$

which we have to solve with the boundary conditions

$$\frac{1}{i}\frac{\delta Z_0}{\delta J_\rho(t)}\bigg|_{t=t_0}=0, \qquad \frac{1}{i}\frac{\delta Z_0}{\delta J^*_\rho(t)}\bigg|_{t=T}=0, \qquad Z_0[\mathbf{0},\mathbf{0}]=1 \tag{B16}$$

Let $D^{(1)}_{\rho\sigma}(t)$ and $D^{(2)}_{\rho\sigma}(t)$ satisfy

$$\begin{aligned}\left[\frac{\partial}{\partial t}\delta^\mu_\rho - B_{\mu\rho}(\boldsymbol{\alpha}(t))\right]D^{(1)}_{\rho\sigma}(t)=0\\ \left[\frac{\partial}{\partial t}\delta^\mu_\rho + B_{\rho\mu}(\boldsymbol{\alpha}(t))\right]D^{(2)}_{\rho\sigma}(t)=0\end{aligned} \tag{B17}$$

with boundary conditions $D^{(j)}_{\rho\sigma}(t=0)=\delta_{\rho\sigma}$, $j=1, 2$. We notice that these functions depend on t through $\boldsymbol{\alpha}(t)$. From (B14) we obtain

$$\frac{1}{i}\frac{\delta Z_0}{\delta J^*_\rho(t)}=-\int_{t_0}^T dt'\, J_\mu(t')S_{\mu\rho}(t',t)\cdot Z_0[\mathbf{J},\mathbf{J}^*] \tag{B18}$$

$$S_{\mu\rho}(t',t)=\theta(t'-t)\,D^{(2)}_{\rho\sigma}(t)\,D^{(2)-1}_{\sigma\mu}(t') \tag{B19}$$

where $D^{(j)-1}$ stands for the inverse matrix of $D^{(j)}$. From (B15) we obtain $[a_{\mu\nu}(\tau)=a_{\mu\nu}(\boldsymbol{\alpha}(\tau))]$

$$\begin{aligned}\frac{1}{i}\frac{\delta Z_0}{\delta J_\rho(t)}=&\left[-\int_{t_0}^T dt'\, S_{\rho\alpha}(t,t')J^*_\alpha(t')\right.\\ &\left.+i\int_{t_0}^T dt'\, \Delta^{\rho\gamma}(t,t')\,J_\gamma(t')\right]\cdot Z_0\end{aligned} \tag{B20}$$

$$\begin{aligned}\Delta^{\rho\gamma}(t,t')=D^{(1)}_{\rho\sigma}(t)&\left[\theta(t-t')\int_{t_0}^{t'} dt''\, D^{(1)-1}_{\sigma\alpha}(t'')\,a_{\alpha\nu}(t'')\,D^{(2)}_{\nu\beta}(t'')\right.\\ &\left.+\theta(t'-t)\int_{t_0}^{t} dt''\, D^{(1)-1}_{\sigma\alpha}(t'')\,\alpha_{\alpha\nu}(t'')\,D^{(2)}_{\nu\beta}(t'')\right]D^{(2)-1}_{\beta\gamma}(t')\end{aligned} \tag{B21}$$

In the derivation of these formulas we have used the matrix relation $D^{(1)-1}=D^{(2)T}$, $D^{(2)-1}=D^{(1)T}$ where the superscript T stands for the transposed matrix [these relations are derived from (B17)]. One can check also that $\Delta^{\rho\gamma}(t,t')=\Delta^{\gamma\rho}(t',t)$. From (B19) and (B20) we obtain

$$\begin{aligned}Z_0[\mathbf{J},\mathbf{J}^*]=\exp&\left[-i\int_{t_0}^T dt_1\,dt_2\,J_\mu(t_1)\,S_{\mu\rho}(t_1,t_2)\,J^*_\rho(t_2)\right.\\ &\left.-\frac{1}{2}\int_{t_0}^T dt_1\,dt_2\,J_\mu(t_1)\,\Delta^{\mu\nu}(t_1,t_2)\,J_\nu(t_2)\right]\end{aligned} \tag{B22}$$

These formulas are valid in general and describe fluctuations around the solution $\boldsymbol{\alpha}(t)$ of the deterministic dynamical system. If we want to study the stationary fluctuations we have again to take the limit $t_0 \to -\infty$ in our formulas. Since the whole expansion depends on $\boldsymbol{\alpha}(t; \boldsymbol{\alpha}^0, t_0) = \bar{\boldsymbol{\alpha}}(\tau, \boldsymbol{\alpha}^0)$, $\tau = t - t_0$, we shall fall in the attractor of the dynamical system $\dot{x}_\mu = a_\mu(\mathbf{x}(t))$, $\mu = 1, 2,..., M$ [see (B12)], to whose basin of attraction $\boldsymbol{\alpha}^0$ belongs. Since we are now in several dimensions ($M > 1$), the result will depend on the nature of the attractor and each case must be studied separately. The case of a strange attractor would be of special interest. We can give simple formulas when the attractor is a locally stable equilibrium point $\boldsymbol{\beta} = (\beta_1,..., \beta_M)$. We can make a translation to locate this point at the origin and furthermore we assume that we can diagonalize the linear part there; then the dynamical system will be of the form

$$\dot{q}_\mu = \lambda_{(\mu)} q_\mu + b_\mu(\mathbf{q}) \tag{B23}$$

with $\lambda_{(\mu)} < 0$, since the origin is an attractor. One has now that

$$D^{(1)}_{\rho\sigma}(t) = \delta_{\rho\sigma} e^{\lambda_{(\rho)} t}, \qquad D^{(2)}_{\rho\sigma}(t) = \delta_{\rho\sigma} e^{-\lambda_{(\rho)} t}$$

and consequently Z_0^{stat} is given by

$$\begin{aligned} Z_0^{\text{stat}}[\mathbf{J}, \mathbf{J}^*] = \exp\Bigg[& -i \int_{-\infty}^{\infty} dt_1\, dt_2\, J_\mu(t_1)\, S^{\text{stat}}_{\mu\rho}(t_1 - t_2)\, J^*_\rho(t_2) \\ & - \frac{1}{2} \int_{-\infty}^{\infty} dt_1\, dt_2\, J_\mu(t_1)\, \Delta^{\mu\nu}_{\text{stat}}(t_1 - t_2)\, J_\nu(t_2) \Bigg] \end{aligned} \tag{B24}$$

$$S^{\text{stat}}_{\mu\rho}(t) = \delta_{\mu\rho} \theta(t)\, e^{\lambda_{(\rho)} t} \tag{B25}$$

$$\begin{aligned} \Delta^{\mu\nu}_{\text{stat}}(t - t') = -\frac{a_{\mu\nu}(\mathbf{0})}{\lambda_{(\mu)} + \lambda_{(\nu)}} & [\theta(t - t')\, e^{\lambda_{(\mu)}(t - t')} \\ & + \theta(t - t')\, e^{\lambda_{(\nu)}(t' - t)}] \end{aligned} \tag{B26}$$

On the other hand,

$$Z^{\text{stat}}[\mathbf{J}, \mathbf{J}^*] = K^{\text{stat}} \left[\frac{1}{i} \frac{\delta}{\delta \boldsymbol{J}^*}, \frac{1}{i} \frac{\delta}{\delta \mathbf{J}} \right] \cdot Z_0^{\text{stat}}[\mathbf{J}, \mathbf{J}^*] \tag{B27}$$

where $K^{\text{stat}}[\mathbf{p}, \mathbf{q}]$ is given by (B11) by putting there $\boldsymbol{\alpha}(t) = 0$. The stationary correlation functions are then given by (B13) with Z replaced by Z^{stat}. The situation in which one cannot linearize as in (B23) due to the appearance of complex eigenvalues or Jordan blocks can be treated in a similar way with some slight changes.

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